General Gauss-Bonnet brane cosmology

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Abstract

We consider 5-dimensional spacetimes of constant 3-dimensional spatial curvature in the presence of a bulk cosmological constant. We find the general solution of such a configuration in the presence of a Gauss-Bonnet term. Two classes of non-trivial bulk solutions are found. The first class is valid only under a fine tuning relation between the Gauss-Bonnet coupling constant and the cosmological constant of the bulk spacetime. The second class of solutions are static and are the extensions of the AdS-Schwarzchild black holes. Hence in the absence of a cosmological constant or if the fine tuning relation is not true, the generalised Birkhoff's staticity theorem holds even in the presence of Gauss-Bonnet curvature terms. We examine the consequences in brane world cosmology obtaining the generalised Friedmann equations for a perfect fluid 3-brane and discuss how this modifies the usual scenario.

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I. INTRODUCTION

The intriguing possibility that our Universe is only part of a higher dimensional spacetime [1], [2] has raised a lot of interest in the physics community recently [3], [4], [5] [6], [7], [8]. In particular 5 dimensional brane Universe models and their cosmology have been extensively studied (see for example [9], [10], [11]). The Universe in this case is a gravitating homogeneous and isotropic brane or domain wall evolving in some constant bulk curvature spacetime.

An interesting feature of such a configuration is that it verifies a generalised version of Birkhoff's staticity theorem [10], [11]: a constant curvature spacetime of constant 3-space curvature is locally static; more specifically an ADS-black hole solution [12],

$$ds^2 = -H(r)dt^2 + H^{-1}(r)dr^2 + \left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega_{II}^2\right) \tag{1}$$

where $H(r) = \kappa - \frac{\mu}{r^2} + k^2 r^2$, with $\kappa = 0, \pm 1$ and μ, k^2 are related to the black hole mass and bulk cosmological constant respectively. To ensure the validity of this theorem it is essential firstly that the brane Universe is of co-dimension 1 i.e. a domain wall type defect and secondly that the brane Universe is homogeneous and isotropic. The theorem in turn implies a certain number of physical properties for the configuration, in particular that the only physical dynamical degree of freedom is the wall's trajectory or equivalently, for the 4-dimensional observer stuck on the brane, the expansion rate of the Universe. Hence although we have introduced an extra dimension, the number of dynamical degrees of freedom does not at all alter with respect to standard 4 dimensional FLRW cosmology. Just like in 4 dimensional cosmology, once given an equation of state relating energy density and pressure one obtains the expansion rate or equivalently the brane Universe trajectory.

Furthermore we know that 4 dimensional gravity is quite special for a numerous number of reasons. For example $D = 4$ gives the minimal number of dimensions where the graviton is non trivial and has exactly two polarisation degrees of freedom whereas at the same time gauge interactions of the Standard Model are renormalisable. Another special property of 4 dimensional gravity is the uniqueness of the Einstein-Hilbert action.

In $D > 4$, however, the situation is quite different. In 5 dimensions, in order to obtain a *unique* action, i.e. giving rise to a second order symmetric and divergence-free tensor, and to field equations that are of second order in the metric components, *we have to add* the Gauss-Bonnet term to the Einstein-Hilbert plus cosmological constant action. This is part of Lovelock's theorem [13]. Furthermore in string theory the Gauss-Bonnet term corresponds to the leading order quantum correction to gravity. Its coupling constant is related to the Regge slope parameter or string scale. Furthermore string theories contain no ghosts. Interestingly as was demonstrated in [14] the only fourth order higher derivative term to give ghost-free self-interactions for the graviton (around flat spacetime) is precisely the Gauss-Bonnet term.

The reason for all these nice properties shared by the Einstein-Hilbert and the Gauss-Bonnet terms can be understood and generalised from a purely geometrical point of view. The Gauss-Bonnet term is the generalised Euler characteristic of a 4 dimensional spacetime. It yields in $D = 4$ a boundary term (topological and not dynamical contribution). This is a quite general and elegant fact. Indeed in a similar fashion the Einstein-Hilbert action is related to the usual Euler characteristic of a 2-dimensional manifold. For example for string field theory the Euler characteristic is related then to the string coupling constant g_s governing "surface diagrams" in the perturbative regime. In general for every even dimension of spacetime we have to add a higher order correction to the gravitational action in order to preserve uniqueness, thus for example in 10 dimensions one has the Euler characteristics of 0 (cosmological constant), 2 (Einstein-Hilbert), 4 (Gauss-Bonnet), 6, and 8 dimensional manifolds [13]. So from this discussion it would seem natural to include the Gauss-Bonnet term in a 5 dimensional spacetime, all the more since we are interested in toy models merging string theory with standard cosmology.

Madore and collaborators have considered this term in order to stabilise the 5th dimension in Kaluza-Klein theories [15] whereas there was a lot of effort in the 80's to obtain exact solutions in Gauss-Bonnet theories in view to their relevance to quantum gravity corrections of string theory (see for instance [16], [17] [18], [19], [20]). More recently in the context of brane Universe cosmology it has been shown that the localised graviton zero mode persists in the RS model [21] in the presence of a Gauss-Bonnet term. Cosmological consequences have also been studied in [22]. However, only particular solutions in the bulk have been considered. Here we shall attack the problem in its full generality. We shall first of all find and discuss the full bulk solutions, and then we shall investigate the brane cosmology they induce. Not surprisingly Birkhoff's theorem will be in the centre of our analysis and its physical consequences.

In the next section we set up the basic ingredients of the problem. In Section III we solve by brute force the field equations and find the general solution for the bulk spacetime. In Section IV we discuss the relevance of the bulk solutions that we find to brane Universe cosmology in 5 dimensions. We conclude in section V.

II. GENERAL SETTING

Consider the following 5-dimensional action,

$$S = \frac{M^3}{2} \int d^5x \sqrt{-g} \left[R + 6k^2 + \alpha (R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) \right] , \tag{2}$$

where M is the fundamental mass scale of the 5-dimensional theory, $\Lambda = -6k^2$ is the negative bulk cosmological constant and the Gauss-Bonnet coupling constant α of dimension $(length)^2$,which we leave free, is the additional physical parameter of the problem. Setting $\alpha = 0$ we obviously get the usual Einstein-Hilbert action with cosmological constant in 5 dimensions. As we discussed in the introduction, just like the Einstein-Hilbert action with cosmological constant is unique in 4 dimensions, the gravitational action (2) is unique in 5 dimensions. For this reason and for clarity we shall restrict ourselves to 5 dimensions.

Let us now consider a spacetime with constant three-dimensional spatial curvature. A general metric can then be written,

$$ds^2 = e^{2\nu(t,z)} B(t,z)^{-2/3} (-dt^2 + dz^2) + B(t,z)^{2/3} \left(\frac{d\chi^2}{1 - \kappa\chi^2} + \chi^2 d\Omega_{II}^2 \right) \tag{3}$$

where $B(t,z)$ and $\nu(t,z)$ are the unknown component fields of the metric and $\kappa = 0, \pm 1$ is the normalised curvature of the 3-dimensional homogeneous and isotropic hypersurfaces. We choose to use the conformal gauge in order to take full advantage of the 2-dimensional conformal transformations in the $t - z$ plane. The field equations we are seeking to solve are found by varying the above action (2) with respect to the background metric and read

$$\begin{aligned} \mathrm{E}_{\mu\nu} = G_{\mu\nu} - 6k^2 g_{\mu\nu} - \alpha \Big[\frac{g_{\mu\nu}}{2} (R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\gamma\delta}R^{\gamma\delta} + R^2) \\ - 2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}{}_{\nu} + 4R_{\gamma\delta}R^{\gamma}{}_{\mu}{}^{\delta}{}_{\nu} - 2R_{\mu\gamma\delta\lambda}R_{\nu}{}^{\gamma\delta\lambda} \Big] = 0 \end{aligned} \quad (4)$$

where now the symmetric tensor $\mathrm{E}_{\mu\nu}$ has replaced the usual Einstein tensor $G_{\mu\nu}$ and is also divergence free. Taking the trace of (4) one can show that for a solution, the Ricci scalar is a multiple of the Lagrangian in (2) (see for instance [19]). Thus the behaviour (in particular singularities) of the scalar curvature R is shared by the Gauss-Bonnet scalar in (2). Hence we can deduce that although the field equations change, spacetime curvature still plays the same physical role.

III. THE GENERAL SOLUTION FOR THE BULK SPACETIME

Before plunging in the field equations[1] it is rather useful to review the generalisation of Birkhoff's theorem in the presence of a cosmological constant as it appeared recently in [11]. Furthermore we shall use exactly the same method to derive the general solution.

In this subcase the field equations are obtained setting $\alpha = 0$ in (2) and read

$$R_{\mu\nu} = -\frac{2\Lambda}{3} g_{\mu\nu}.$$

There are two key ingredients in this method. First of all in order to make use of the $t - z$ conformal symmetries of (3) it is important to pass to light-cone coordinates,

$$u = \frac{t-z}{2}, \qquad v = \frac{t+z}{2}. \quad (5)$$

[1] For the full field equations see Appendix.

Secondly taking the combination $R_{tt}+R_{zz}\pm 2R_{tz}=0$, one obtains the integrability conditions which read[2]

$$B_{,uu}-2\nu_{,u}B_{,u}=0, \tag{6}$$

$$B_{,vv}-2\nu_{,v}B_{,v}=0. \tag{7}$$

Note then that these are ordinary differential equations with respect to u and v respectively and are independent of the physical parameter of the problem, namely, the cosmological constant Λ. As their name indicates they are directly integrable giving

$$B=B(U+V) \qquad e^{2\nu}=B'U'V' \tag{8}$$

where $U=U(u)$ and $V=V(v)$ are arbitrary functions of u and v, and a prime stands for the total derivative of the function with respect to its unique variable. Using a conformal transformation,

$$U=\frac{\tilde{z}-\tilde{t}}{2}, \qquad V=\frac{\tilde{z}+\tilde{t}}{2}$$

gives that the solution is locally static $B=B(\tilde{z})$ and Birkhoff's theorem is therefore true. Note that we did not have to find the precise form of the solution for B. The integrability conditions actually suffice to prove staticity and thus Birkhoff's theorem. By use of the remaining field equations we can then find the form of B, leading after coordinate transformation to the AdS-Schwarchild black hole solution (1). Note that the solution becomes $\tilde{t}$-dependent as we cross the event horizon of the black hole.

Let us now go to our case of interest with $\alpha\neq 0$. In the presence of the Gauss-Bonnet term we can expect that if the system is indeed integrable then some integrability equation should be reproduced. Putting away technicalities this is the essence of what we shall do here. In analogy to the previuos case let us take the combination, $\mathrm{E}_{tt}+\mathrm{E}_{zz}\pm 2\mathrm{E}_{tz}=0$. On passing to light cone coordinates (5) we get after some manipulations

[2]From now on $B_{,u}$ represents the partial derivative of B with respect to u.

$$\left(9B^{4/3}e^{2\nu}+36\alpha\kappa B^{2/3}e^{2\nu}+4\alpha B_{,u}B_{,v}\right)\left(B_{,uu}-2\nu_{,u}B_{,u}\right)=0$$
$$\left(9B^{4/3}e^{2\nu}+36\alpha\kappa B^{2/3}e^{2\nu}+4\alpha B_{,u}B_{,v}\right)\left(B_{,vv}-2\nu_{,v}B_{,v}\right)=0 \tag{9}$$

Note how the Gauss-Bonnet terms factorise nicely leaving the integrability equations (6) we had in the absence of α.

Let us neglect for the moment the degenerate case where either $B_{,u}=0$ or $B_{,v}=0$ which corresponds to flat solutions [23]. For $B_{,u}\neq 0$ or $B_{,v}\neq 0$ the situation is clear: either we have static solutions and Birkhoff's theorem holds as in the case above or we will have

$$e^{2\nu}=\frac{4\alpha(B_{,z}^2-B_{,t}^2)}{9B^{2/3}(B^{2/3}+4\alpha\kappa)} \tag{10}$$

Let us first examine the latter case, that we will call Class I solution. The two remaining wave equations $E_{\chi\chi}=0$ and $E_{tt}-E_{zz}=0$ give after some algebra the simple relation,

$$8\alpha k^2=1 \tag{11}$$

This is quite remarkable: if the coupling constants obey this simple relation (11) then the B field is an *arbitrary* function of space and time. Note in passing that Class I solutions exist in arbitrary dimension d if the fine tuning relation $\frac{96\alpha k^2}{(d-1)(d-2)}=1$ is satisfied. We can already deduce that Birkhoff's theorem does not hold for non zero cosmological constant. [3] However in the absence of a cosmological constant it is always trivially true since (11) is impossible. Also we can note that a positive Gauss-Bonnet constant $\alpha>0$, as in heterotic string theory, demands a negative cosmological constant and vice-versa. The Class I metric reads,

$$ds^2=\frac{4\alpha(B_{,z}^2-B_{,t}^2)}{9B^{4/3}(B^{2/3}+4\alpha\kappa)}(-dt^2+dz^2)+B^{2/3}\left(\frac{d\chi^2}{1-\kappa\chi^2}+\chi^2d\Omega_{II}^2\right) \tag{12}$$

under the constraint (11) where we emphasize that $B(t,z)$ is an

arbitrary function of t and z. To simplify somewhat set $B=R^3$ to get,

[3] Note however that for a non-zero charge Q and spherical symmetry ($\kappa=1$) Birkhoff's theorem is always true as was shown by Wiltshire [17] (see also [18])

$$ds^2 = \frac{R_{,z}^2 - R_{,t}^2}{\kappa + \frac{R^2}{4\alpha}}(-dt^2 + dz^2) + R^2 \left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega_{II}^2 \right) \tag{13}$$

This solution has generically a curvature singularity for $R_{,z}^2 = R_{,t}^2$. The parameter α is related here to the 5-dimensional cosmological constant via (11). The Class I static solutions are given by,

$$ds^2 = -\frac{A(R)^2}{\kappa + \frac{R^2}{4\alpha}} dt^2 + \frac{dR^2}{\kappa + \frac{R^2}{4\alpha}} + R^2 \left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega_{II}^2 \right) \tag{14}$$

with $A = A(R)$ now an arbitrary function of R. Time-dependent solutions for $\alpha > 0$ are only possible for $R^2 < 4\alpha$ and $\kappa = -1$.[4]

In order to obtain t and z dependent solutions it suffices to take the functional R to be a non-harmonic function. Take for instance $R = exp(f(t) + g(z))$, with f and g arbitrary functions. Let us also assume $\kappa = 0$ for simplicity, the Class I metric in proper time reads,

$$ds^2 = -d\tau^2 + \frac{4\alpha dg^2}{1 + 4\alpha f_{,\tau}^2} + e^{2(f+g)} \left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega_{II}^2 \right). \tag{15}$$

On the other hand if (11) does not hold then Birkhoff's theorem remains true in the presence of the Gauss-Bonnet terms i.e. the general solution assuming the presence of a cosmological constant in the bulk and 3 dimensional constant curvature surfaces is static if and only if (11) is not satisfied. In this case the remaining two equations give the same ordinary differential equation for $B(U+V)$ which after one integration reads,

$$B' + 9B^{2/3}(k^2 B^{2/3} + \kappa) + 18\alpha \left(\frac{B'}{9B^{2/3}} + \kappa \right)^2 = 9\mu \tag{16}$$

where μ is an arbitrary integration constant. Then by making B the spatial coordinate and setting $B^{1/3} = r$ we get the solution discovered and discussed in detail by Boulware-Deser [16] ($\kappa = 1$) and Cai [24] ($\kappa = 0, -1$),[5]

$$ds^2 = -V(r) dt^2 + \frac{dr^2}{V(r)} + r^2 \left(\frac{d\chi^2}{1-\kappa\chi^2} + \chi^2 d\Omega_{II}^2 \right) \tag{17}$$

[4] For $\alpha < 0$ the situation is interchanged with static solutions possible only for $\kappa = 1$ and $R^2 < -4\alpha$.

[5] We have kept the same label as in (3) for the rescaled time coordinate .

where $V(r) = \kappa + \frac{r^2}{4\alpha}[1 \pm \sqrt{1 - 8\alpha k^2 + 8\frac{\alpha\mu}{r^4}}]$, and μ plays the role of the gravitational mass. The maximally symmetric solutions are obtained by setting $\mu = 0$. There are two AdS branches permitted by the solution for $\alpha > 0$ ([24], [16]). We can have both de-Sitter and Anti-de-Sitter for $\alpha < 0$. Not surprisingly, as shown by Boulware and Deser [16], only one of the branches is physical, the $'+'$ branch being classically unstable to small perturbations and yielding a graviton ghost. For $\alpha > 0$ and the $'-'$ branch (which we consider from now on) there is a black hole singularity at $r = 0$, a unique event horizon and asymptotically one approaches the 5-d Schwarzchild solution for $\kappa = 1$ (for a general and thorough analysis of these black hole solutions, their stability and their thermodynamics we refer the reader to [24], [16]).

Now notice how (11) is a particular point for (17) since the maximally symmetric solution is defined for $1 \geq 8\alpha k^2$. In fact when (11) is satisfied and $\mu = 0$, the two branches coincide and $V = \kappa + \frac{r^2}{4\alpha}$, which coincides with (12) for the particular value of $A = V$.

Furthermore for small α we have, $V(r) = \kappa + k^2 r^2 (1 + 2\alpha k^2) - \frac{\mu}{r^4}[1 + 2\alpha(2k^2 - \frac{\mu}{r^4})] + O(\alpha^2)$ and indeed for $\alpha = 0$ we get the usual Kottler solution [12].

IV. BRANE WORLD COSMOLOGY

Having evaluated the general solution in the bulk we now consider a 4-dimensional 3-brane where matter is confined. We furthermore suppose following the metric symmetries (3) that matter on the brane is a perfect fluid of energy density ρ and pressure p. We consider the brane fixed at $z = 0$ and the bulk spacetime evolving in time. By virtue of Birkhoff's theorem this is equivelant to taking a moving brane in the static black hole background (17).

The energy-momentum brane tensor takes the form,

$$T_{\mu}^{\nu(b)} = \frac{\delta(z)}{\sqrt{g_{zz}}} diag(-\rho(t), p(t), p(t), p(t), 0).$$

The field equations read,

$$\mathrm{E}_{\mu\nu} = M^{-3} T_{\mu\nu}^{(b)} e^{-\nu(t,0)} B^{1/3}(t,0)\delta(z). \tag{18}$$

We assume Z_2 symmetry and set $M^3 = 1$ for the moment. Now before proceeding there are two important points to take into account. First of all the Israel junction conditions are no longer valid since we have included Gauss-Bonnet terms in the gravitational action. Although the Gauss-Codazzi integrability conditions are universal for any spacelike or timelike hypersurface, the Israel junction conditions have to be generalised in order to take into account the addition of the Gauss-Bonnet terms [25] in the gravitational action (2). So in order to evaluate the brane junction conditions we bifurcate the Israel junction conditions matching the distributional part of the field equations (18). Secondly since the field equations are of second order we will encounter at most second derivatives of z and therefore the metric component fields have to be continuous [6]. Indeed first order derivatives contain a jump in the metric given by means of the Heaviside distribution whereas second order derivatives contain a Dirac distribution at $z = 0$ as they should. Note that there are no ill-defined distributional products appearing in the field equations, since first derivatives (multiplying second derivatives) are always encountered as squares, eliminating thus the Heaviside distributions [22]. This however would not have been the case for any other quadratic curvature terms in the bulk action, but rather is another interesting property of the Gauss-Bonnet combination. This is coherent with the fact that since (2) is unique we expect a regular gravity theory in 5 dimensions and hence regular boundary conditions.

Let us first check the boundary conditions for the fine-tuned Class I solutions. The integrability conditions read

$$I_1(B_{,uu} - 2\nu_{,u}B_{,u}) \sim \rho\delta(z)$$

$$I_1(B_{,vv} - 2\nu_{,v}B_{,v}) \sim \rho\delta(z)$$

Obviously if $I_1 = 9B^{4/3}e^{2\nu} + 36\alpha\kappa B^{2/3}e^{2\nu} - 4\alpha(B_{,z}^2 - B_{,t}^2) = 0$ then $\rho = 0$ (using the $\chi - \chi$ field equation we can find that $p = 0$, see appendix). Suppose now that $I_1 \neq 0$ at

[6]Note that had we considered higher than second derivative contributions to the field equations the situation would have been totally different

$z = 0$, whereas $I_1 = 0$ elsewere in the bulk. Then from (19) we have $B_{,zz} \sim \delta(z)$, hence B is continuous and we are basically led to a contradiction since then $I_1(z=0)=0$. Therefore Class I solutions cannot hold distributional brane boundaries. This stems from the fact that I_1 does not contain second order derivatives. Hence the presence of Dirac branes of localised matter demands static solutions (17), which in turn guarantees from Birkhoff's theorem that there is only one dynamical degree of freedom, as in 4-dimensional FLRW cosmology, the brane trajectory.

For the black hole solutions (17) we have $I_1 \neq 0$. Therefore the metric components are now continuous across the brane with the second derivatives giving a Dirac distribution at $z = 0$. Matching the δ distributions and assuming Z_2 symmetry we generically obtain,

$$\rho = -\frac{2B_{,z}I_1}{9e^{3\nu}B^2} = -\frac{B'(V'-U')}{e^{\nu}B^{2/3}}\left[1+4\alpha(\kappa B^{-2/3}+\frac{B'}{9B^{4/3}})\right] \qquad (19)$$

$$p = \frac{2}{9e^{3\nu}B^2}\left[B_{,z}[12e^{2\nu}B^{2/3}(2\alpha\kappa+B^{2/3})-I_1+8\alpha B(B_{,tt}-\nu_{,t}B_{,t}-\nu_{,z}B_{,z})]+I_1\nu_{,z}B)\right] \quad (20)$$

It is interesting to note that the junction conditions remain invariant under the conformal boost $u \to f(u)$, $v \to f(v)$ just as for $\alpha = 0$. This in essence means that there is a single degree of freedom U' or V' which coordinate transforming corresponds to the wall's trajectory in the black hole background (17) (see [11] for a detailed discussion).

Consider now a brane Universe observer. The expansion parameter (or wall's trajectory) reads, $R(\tau) = B^{1/3}(t,0)$ whereas proper time τ is given by $d\tau = e^{\nu(t,0)}B^{-1/3}(t,0)dt$. For the Class II solutions we have (8) and for example the Hubble expansion rate is given by,

$$H = \frac{1}{R}\frac{dR}{d\tau} = \frac{(U'+V')B'}{6e^{\nu}B^{2/3}}$$

First, using (16), (19) and (20), one may check that one still has the standard conservation equation on the brane:

$$\frac{d\rho}{d\tau} + 3H(p+\rho) = 0$$

which is a result of the Bianchi and Bach-Lanczos identities for the Einstein tensor and Gauss-Bonnet terms respectively in (4). Then, using (16) and (19) we get the generalised Friedmann equation,

$$H^2 = \frac{\rho^2}{36}(1 - 8\alpha k^2 + \frac{8\alpha\mu}{R^4})^{-1} - \frac{\kappa}{R^2} - \frac{1}{4\alpha}\left[1 \pm \sqrt{1 - 8\alpha k^2 + \frac{8\alpha\mu}{R^4}}\right] \tag{21}$$

We can recast this equation in terms of the black hole potential,

$$3H^2 = \frac{\rho^2}{12}(1 + C_1)^{-1} - \frac{V(r)}{R^2}$$

with

$$C_1 = -8\alpha k^2 + \frac{8\alpha\mu}{R^4}$$

and note that the C_1 term is the extra term in the Israel junction conditions. Note already that the $+$ branch of the black hole solution yields a singular negative term for $\alpha \to 0$. This ties in nicely with the results of Boulware and Deser who showed that this branch was unphysical (unstable). From here on we consider the $-$ branch only. One can similarily obtain using (16), (19) and (20) the acceleration equation,

$$\begin{aligned}\frac{R_{,\tau\tau}}{R} = &-\frac{\rho}{36}\left[3p + 2\rho\frac{1 - 8\alpha k^2}{1 - 8\alpha k^2 + \frac{8\alpha\mu}{R^4}}\right]\left(1 - 8\alpha k^2 + \frac{8\alpha\mu}{R^4}\right)^{-1} \\ &- \frac{2\mu}{R^4}\left(1 - 8\alpha k^2 + \frac{8\alpha\mu}{R^4}\right)^{-1/2} - \frac{1}{4\alpha}\left[1 - \left(1 - 8\alpha k^2 + \frac{8\alpha\mu}{R^4}\right)^{1/2}\right]\end{aligned} \tag{22}$$

If we consider the limit of small Gauss Bonnet coupling constant, equations (21) and (22) reduce to,

$$H^2 = \frac{\rho^2}{36} - \frac{\kappa}{R^2} - k^2 + \frac{\mu}{R^4} + 2\alpha(k^2 - \frac{\mu}{R^4})(\frac{\rho^2}{9} - k^2 + \frac{\mu}{R^4}) + O(\alpha^2), \tag{23}$$

$$\begin{aligned}\frac{R_{,\tau\tau}}{R} = &-\frac{\rho}{36}(2\rho + 3p) - k^2 - \frac{\mu}{R^4} - \frac{2\alpha}{9}\rho(2\rho + 3p)(k^2 - \frac{\mu}{R^4}) + \frac{4\alpha}{9}\rho^2\frac{\mu}{R^4} \\ &- 2\alpha(k^2 - \frac{\mu}{R^4})(k^2 + \frac{3\mu}{R^4}) + O(\alpha^2)\end{aligned} \tag{24}$$

where the first order corrections to the usual generalised FRLW from a 5D theory (see for instance [10]) appear now clearly. The Gauss-Bonnet parameter couples to the remaining black hole parameters and to the energy density of the brane.

To study cosmology at late times, we will take into account the tension (vacuum energy) T of the brane ($\rho \to T + \rho$ and $p \to p - T$), keeping linear terms in ρ, p, and $\frac{\alpha\mu}{R^4}$ (large scale factor). Then for a zero cosmological constant on the brane (critical case) one has to impose in (21) and (22) a modified Randall-Sundrum condition

$$T = 3\left[\frac{1}{\alpha}(1-\sqrt{1-8\alpha k^2})(1-8\alpha k^2)\right]^{\frac{1}{2}} \tag{25}$$

which indeed allows for a Kaluza-Klein zero-mode localized on the brane [21]. In this case (21) and (22) give,

$$H^2 = \frac{1}{6}\left[\frac{1-\sqrt{1-8\alpha k^2}}{\alpha(1-8\alpha k^2)}\right]^{1/2}\rho - \frac{\kappa}{R^2} + \frac{\mu}{R^4}\left[\frac{3}{\sqrt{1-8\alpha k^2}} - \frac{2}{(1-8\alpha k^2)}\right] \tag{26}$$

$$\frac{R_{,\tau\tau}}{R} = -\frac{1}{12}\left[\frac{1-\sqrt{1-8\alpha k^2}}{\alpha(1-8\alpha k^2)}\right]^{1/2}(\rho+3p) - \frac{\mu}{R^4}\left[\frac{3}{\sqrt{1-8\alpha k^2}} - \frac{2}{(1-8\alpha k^2)}\right] \tag{27}$$

The Gauss-Bonnet term for gravity in the bulk has then essentially two effects on late time brane cosmology: it modifies the black hole term in $\frac{\mu}{R^4}$ and the 4-dimensional Planck mass which is now given by

$$m_{Pl}^{-2} = M^{-3}\,\frac{1-\sqrt{1-8\alpha k^2}}{2\alpha\sqrt{1-8\alpha k^2}} \tag{28}$$

where the fundamental mass scale of the 5-dimensional theory has been restored. Note that these two quantities become divergent in the limit $8\alpha k^2 \to 1$, which is definitively not physical in this context. As usual, the black hole mass term is constrained by Nucleosynthesis (see [9]), but the black hole mass μ itself may now be less constrained thanks to the α-dependent additional factor in the generalized Friedmann equations. For instance, if $\alpha k^2 = 5/72$, the term in μ drops out, leaving the black hole mass μ free, but in this case the 4-dimensional Planck mass and the 5-dimensional fundamental mass scale are of the same order of magnitude. Finally, for $0 < 8\alpha k^2 < 1$, which is the case for known solutions in string theory, one sees from (28) that:

$$M^3 > m_{Pl}^2 k$$

whereas strict equality holds in the absence of the Gauss-Bonnet term (which is the leading quantum gravity correction term). Hence for fixed 4-dimensional Planck mass and cosmological constant in the bulk, “quantum corrections” for gravity in the bulk increase the fundamental mass scale M of the 5-dimensional theory.

If the criticality condition (25) is not satisfied then one has an effective cosmological constant on the brane,

$$\Lambda_{eff} \sim \frac{T^2}{36} - k^2 + 2\alpha k^2 (\frac{T^2}{9} - k^2) + O(\alpha k^2) \tag{29}$$

Note then that the usual Randall-Sundrum criticality condition, $T = 6k$, gives the leading contribution, and if satisfied one obtains a small cosmological constant contribution $\Lambda_{late} \sim \alpha k^4$. If $M < 10^3 TeV$ one is in agreement with the late cosmological constant observations. This could be a hint that string corrections can provide an explanation for the late acceleration of our Universe.

V. CONCLUSIONS

In this paper we have studied Gauss-Bonnet brane cosmology. Our main motivation for including the Gauss-Bonnet term is that the usual 5 dimensional gravitational action (2) is then unique [13] as we noted in the introduction. Furthermore the Gauss-Bonnet coupling constant α provides a window to the leading quantum gravity correction coming from string theory.

Starting from a homogeneous and isotropic 3-space in constant bulk curvature we found the general spacetime solutions to the field equations. Under a particular relation between the bulk cosmological constant and the Gauss-Bonnet coupling, a space and time dependant solution (Class I) of the field equations was found. If however this special relation is not satisfied then the unique solution is the black hole solution discovered and discussed in [16], [19], [17] and [24]. As far as brane Universe cosmology is concerned Class I solutions turn out not to be physical. Therefore quite elegantly Birkhoff's staticity theorem holds and all its interesting properties go through.

On deriving the generalised Friedmann equations we encountered no ill defined distributional products which actually turns out to be yet another nice feature for the Gauss-Bonnet combination. Gauss-Bonnet corrections tend to soften Nucleosynthesis constraints on the black hole mass and to increase the 5 dimensional fundamental mass scale. This may be interesting if we interpret the Gauss Bonnet term as the leading string quantum gravity correction. Furthermore if we consider a departure from criticality i.e. non-zero effective cosmological constant on the brane, then Gauss-Bonnet corrections can yield a small late cosmological constant under the typical RS conditions. We should note here that unfortunately the GB terms do not help at all in justifying the fine tuning conditions necessary to obtain a sensible late time FLRW cosmology.

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Note added: [26] has been brought to our attention, which appeared quasi-simultaneously to our paper and discusses some related issues.

APPENDIX: FIELD EQUATIONS

The field equations obtained from (4) read,

$$\begin{aligned}
\mathrm{E}_{\chi\chi} = & -\frac{1}{9e^{4\nu}B}(B_{,tt}-B_{,zz})[12e^{2\nu}B^{2/3}(2\alpha\kappa+B^{2/3})-I_1] \\
& + \frac{8\alpha}{9e^{4\nu}}[B_{,zz}B_{,tt}+(\nu_{,t}^2-\nu_{,z}^2)(B_{,t}^2-B_{,z}^2)+B_{,tz}(2\nu_{,t}B_{,z}+2B_{,t}\nu_{,z}-B_{,tz}) \\
& - (B_{,tt}+B_{,zz})(\nu_{,t}B_{,t}+\nu_{,z}B_{,z})] \\
& - \frac{20\alpha}{81e^{4\nu}B^2}(B_{,t}^2-B_{,z}^2)^2-\frac{I_1}{9e^{4\nu}}(\nu_{,tt}-\nu_{,zz}) \\
& - 6k^2B^{2/3}-\frac{\kappa}{3e^{2\nu}B^{4/3}}[I_1-6e^{2\nu}B^{2/3}(6\alpha\kappa+B^{2/3})]=0
\end{aligned} \tag{A1}$$

$$\mathrm{E}_{tt}-\mathrm{E}_{zz}=\frac{I_1}{9e^{2\nu}B^{7/3}}(B_{,tt}-B_{,zz})+\frac{12e^{2\nu}k^2}{B^{2/3}}$$

$$+\frac{6\kappa e^{2\nu}}{B^{4/3}}-\frac{8\alpha}{27B^{10/3}e^{2\nu}}(B_{,t}^2-B_{,z}^2)^2$$
$$-\frac{8\alpha\kappa}{3B^{8/3}}(B_{,t}^2-B_{,z}^2)=0 \qquad \text{(A2)}$$

The integrability conditions $\mathrm{E}_{tt}+\mathrm{E}_{zz}\pm 2\mathrm{E}_{tz}$ are:

$$I_1(B_{,tt}+B_{,zz}+2B_{,tz}-2\nu_{,t}B_{,t}-2\nu_{,z}B_{,z}+2\nu_{,t}B_{,z}+2B_{,t}\nu_{,z})=0 \qquad \text{(A3)}$$

$$I_1(B_{,tt}+B_{,zz}-2B_{,tz}-2\nu_{,t}B_{,t}-2\nu_{,z}B_{,z}-2\nu_{,t}B_{,z}-2B_{,t}\nu_{,z})=0 \qquad \text{(A4)}$$

where $I_1=9B^{4/3}e^{2\nu}+36\alpha\kappa B^{2/3}e^{2\nu}-4\alpha(B_{,z}^2-B_{,t}^2)$.

In the degenerate case where either $B_{,u}=0$ or $B_{,v}=0$ (Class I solutions according to Taub) there are now two subcases. Either we obtain the Class I solution of Taub [23] which is simply flat Minkowski spacetime or we obtain,

$$ds^2=e^{2\nu}(-4\alpha\kappa)^{-1}(-dt^2+dz^2)+(-4\alpha\kappa)\frac{d\chi^2}{1-\kappa\chi^2}+\chi^2 d\Omega_{II}^2) \qquad \text{(A5)}$$

under once again (11). Note once more that $\nu(t,z)$ is an arbitrary function of t and z and planar symmetry is not permitted.

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